A Survey of Recent Results on Key-Alternating Ciphers

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(based on joint work with R. Lampe and J. Patarin)

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A key-alternating cipher with \( r \) rounds is the following construction:

\[
x \xrightarrow{\gamma_0} K \xrightarrow{P_1} K \xrightarrow{\gamma_1} P_2 \xrightarrow{\gamma_r} K \xrightarrow{P_r} y
\]

- The \( P_i \)'s are public permutations on \( \{0, 1\}^n \)
- \( K \in \{0, 1\}^\ell \) is the (master) key
- The \( \gamma_i \)'s are key derivation functions mapping \( K \) to \( n \)-bit values

Also named Iterated Even-Mansour (IEM) cipher
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\begin{align*}
K & \quad \gamma_0 \quad K \\
\gamma_0 & \quad P_1 \quad \gamma_1 \quad P_2 \quad \ldots \quad P_r \\
x & \quad \oplus & \quad P_1 & \quad \oplus & \quad P_2 & \quad \ldots & \quad P_r & \quad \oplus & \quad y \\
\end{align*}
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Also named Iterated Even-Mansour (IEM) cipher
Most (if not all) SPN ciphers can be described as key-alternating ciphers. E.g. for AES-128, one has \( r = 10 \), the \( \gamma_i \)'s are efficiently invertible permutations, and:

\[
P_1 = \ldots = P_9 = \text{SubBytes} \circ \text{ShiftRows} \circ \text{MixColumns}
\]
\[
P_{10} = \text{SubBytes} \circ \text{ShiftRows}
\]

When the \( P_i \)'s are fixed permutations, one can prove results like:

- the best differential characteristic over \( r' < r \) rounds has probability at most \( p \)
- the best linear approximation over \( r' < r \) rounds has probability at most \( p' \)

This gives upper bounds on the success probability of very specific adversaries.
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- the best linear approximation over $r' < r$ rounds has probability at most $p'$

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Recently, a lot of results have been obtained in the Random Permutation Model: the $P_i$’s are viewed as oracles to which the adversary can make black-box queries (both to $P_i$ and $P_i^{-1}$).

Interpretation: gives a guarantee against any adversary which do not use particular properties of the $P_i$’s

In fact, this model was already considered 15 years ago by Even and Mansour for $r = 1$ round: they showed that the following cipher is secure up to $O(2^{n/2})$ queries of the adversary:
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Outline

1. Indistinguishability
   - Introduction
   - The coupling technique
   - The indistinguishability proof

2. Interlude: tweakable block ciphers

3. Indifferentiability
   - Introduction
   - Indifferentiability of the IEM cipher
   - At least 4 rounds are necessary
   - Indifferentiability proof for 12 rounds
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The IEM cipher with independent keys

We focus in this part on the IEM cipher with independent round keys:

\[ K = (k_0, k_1, \ldots, k_r) \]

Total key space: \( \{0, 1\}^{(r+1)n} \)

Notation:

\[ y = \text{EM}^{P_1, \ldots, P_r}_{(k_0, \ldots, k_r)}(x) \]
Formalizing indistinguishability for the IEM cipher

- left: \( k_0, \ldots, k_r \leftarrow \{0, 1\}^n \) are randomly chosen keys
- right: \( Q \) is a random permutation independent of \( P_1, \ldots, P_r \)
- we are in the Random Permutation Model: the distinguisher also has oracle access to \( P_1, \ldots, P_r \) in both worlds
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Indistinguishability of the IEM cipher: Summary of results

Results for independent round keys \((k_0, k_1, \ldots, k_r)\)

Notation: \(N = 2^n\)

- for \(r = 1\) round, EM is secure up to \(O(N^{1/2})\) queries [EM97]
- for \(r \geq 2\), EM is secure up to \(O(N^{2/3})\) queries [BKL+12]
- for any even \(r\), EM is secure up to \(O(N^r/(r+2))\) queries [LPS12]
- **tight result**: EM is secure up to \(O(N^r/(r+1))\) queries [CS14]

In the following, we focus on the [LPS12] result which uses the coupling technique.
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**Coupling: definition**

**Definition (Coupling)**

Let $\mu$ and $\nu$ be two probability distributions on $\Omega$. A coupling of $\mu$ and $\nu$ is a probability dist. $\lambda$ on $\Omega \times \Omega$ such that:

$$\forall x \in \Omega, \sum_{y \in \Omega} \lambda(x, y) = \mu(x)$$

$$\forall y \in \Omega, \sum_{x \in \Omega} \lambda(x, y) = \nu(y)$$

In other words, $\lambda$ is a joint probability distribution whose marginal distributions are resp. $\mu$ and $\nu$.

**Definition (Statistical distance)**

$$\|\mu - \nu\| = \frac{1}{2} \sum_{x \in \Omega} |\mu(x) - \nu(x)|$$
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The coupling lemma

Lemma

Let \( \mu \) and \( \nu \) be two probability distributions and \( \lambda \) be a coupling. Let \( (X, Y) \sim \lambda \). Then:

\[
\|\mu - \nu\| \leq \Pr[X \neq Y].
\]

- Introduced by Aldous, key tool to study the mixing time of Markov chains
- First used in crypto by Mironov [Mir02] to analyze the shuffle of RC4, later by [MRS09, HR10] to analyze Feistel ciphers
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A (very) simple example

Two couplings of the uniform distribution on \( \{1, 2, 3, 4\} \) with itself:

\[
\begin{array}{cccc}
X/Y & 1 & 2 & 3 & 4 \\
1 & 1/16 & 1/16 & 1/16 & 1/16 \\
2 & 1/16 & 1/16 & 1/16 & 1/16 \\
3 & 1/16 & 1/16 & 1/16 & 1/16 \\
4 & 1/16 & 1/16 & 1/16 & 1/16 \\
\end{array}
\]

\[
\begin{array}{cccc}
X/Y & 1 & 2 & 3 & 4 \\
1 & 1/4 & 0 & 0 & 0 \\
2 & 0 & 1/4 & 0 & 0 \\
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\text{Pr}[X \neq Y] = 3/4
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\text{Pr}[X \neq Y] = 0
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Not all couplings give good upper bounds on \( \|\mu - \nu\| \)

NB: there always exists a coupling \( \lambda \) for which equality

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is achieved (but it may be hard to describe when \( \mu \) and \( \nu \) are not efficiently computable)
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Two coins:

- a perfect one: $p_{\text{head}} = 0.5$
- a biased one: $p'_{\text{head}} = 0.6$

Show that over $N$ tosses, the probability that the biased coin makes $k$ heads is larger than the probability that the perfect coin makes $k$ heads (for any $k \leq N$). Two solutions:

1. compute the binomial law: a bit tedious...
2. couple the two distributions as follows:
   - toss the perfect coin
   - if the perfect coin makes head, the biased coin makes head
   - if the perfect coin makes tail, the biased coin makes head with proba 0.2

$\Rightarrow$ the marginal distributions are correct (simple)
$\Rightarrow$ for any $k$, the biased coin makes $k$ heads with larger probability than the perfect coin (trivial)
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Two types of distinguishers

NB: $\mathcal{D}$ is computationally unbounded and makes at most $q$ queries to each oracle

We define the two following classes of distinguishers:

- **NCPA** (Non-Adaptive Chosen Plaintext Attacks):
  → works in two phases:
    - $\mathcal{D}$ first queries $P_1,\ldots,P_r$ as it wishes (in both directions, adaptively);
    - then it makes $q$ non-adaptive direct queries to $E_{P_1,\ldots,P_r}/Q$

- **CCA** (Chosen Ciphertext Attacks):
  → the most general class of distinguisher, can adaptively query all oracles in both directions, in any order
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The case of NCPA distinguishers: the result

We will show the following:

**Theorem**

*For any NCPA \( \mathcal{D} \) making at most \( q \) queries to each oracle, the distinguishing advantage against the IEM with \( r \) rounds is at most*

\[
2^r \frac{q^{r+1}}{N^r}.
\]

\( \rightarrow \) security up to \( \mathcal{O}(N^r/(r+1)) \) queries.
The case of NCPA distinguishers: a matching attack

security up to $\mathcal{O}(N^{r/(r+1)})$ queries.

A matching attack has been described in [BKL$^+$12]:

- make $\mathcal{O}(N^{r/(r+1)})$ queries to the cipher and to each $P_i$
- for each possible key, find a "contradictory path"
- any wrong key will have a contradictory path with high proba.
- (note: this is just exhaustive key search, but we are interested in the number of queries rather than computational cost)
The case of NCPA distinguishers

\[ D \text{ first makes } q \text{ queries to } P_1, \ldots, P_r \text{ and obtains equations:} \]

\[ P_i(a_{i,j}) = b_{i,j}, \quad i \in [1, r], \quad j \in [1, q] \ . \]

Then it makes \( q \) non-adaptive queries \((x_1, \ldots, x_q)\) to \( EM/Q \) and receives answers \((y_1, \ldots, y_q)\)
The case of NCPA distinguishers

The distribution of \((a_{i,j}), (b_{i,j})\) is the same in both worlds
→ the advantage of \(D\) is given by the statistical distance between the distributions of \((y_1, \ldots, y_q)\) in the real and the ideal world

Notation:
\[
\mu_q = \text{distribution of } (y_0, \ldots, y_q) \text{ in the real world}
\]
\[
\mu_0 = \text{distribution of } (y_0, \ldots, y_q) \text{ in the ideal world (uniform)}
\]
→ we want to upper bound \(\|\mu_q - \mu_0\|\)
The case of NCPA distinguishers

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→ we want to upper bound \(\|\mu_q - \mu_0\|\)
The case of NCPA distinguishers

The distribution $\mu_q$ in the real world is obtained as follows:

- draw random permutations $P_1, \ldots, P_r$ satisfying $P_i(a_{i,j}) = b_{i,j}$
- draw independent random round keys $(k_0, \ldots, k_r)$
- let $y_i = EM_{P_1, \ldots, P_r}^{(k_0, \ldots, k_r)}(x_i)$
A hybrid argument

\[
\begin{align*}
\text{Distrib. } \mu_q \\
(x_1, \ldots, x_q) \\
\xrightarrow{\text{EM}^{P_1, \ldots, P_r}_{(k_0, \ldots, k_r)}} \\
(y_1, \ldots, y_q)
\end{align*}
\]

\[
\begin{align*}
\text{Distrib. } \mu_0 \\
(x_1, \ldots, x_q) \\
\xrightarrow{Q} \\
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The uniform distribution \(\mu_0\) is also obtained by drawing uniformly random (distinct) inputs \((u_1, \ldots, u_q)\) and computing their image through \(\text{EM}\).
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\[ \text{Distrib. } \mu_q \]
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A hybrid argument

Hybrid distributions $\mu_{\ell}$, $\ell \in [0, q]$

$$\|\mu_q - \mu_0\| \leq \sum_{\ell=0}^{q-1} \|\mu_{\ell+1} - \mu_{\ell}\| .$$

→ We will upper bound $\|\mu_{\ell+1} - \mu_{\ell}\|$ with a coupling.
A hybrid argument

Hybrid distributions $\mu_\ell$, $\ell \in [0, q]$

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Coupling $\mu_{\ell+1}$ and $\mu_{\ell}$

$\text{Distrib. } \mu_{\ell+1}$

$$(x_1, \ldots, x_{\ell}, x_{\ell+1}, u_{\ell+2}, \ldots, u_q)$$

$$\xrightarrow{\text{EM}^{P_1,\ldots,P_r}_{(k_0,\ldots,k_r)}}$$

$$(y_1, \ldots, y_{\ell}, y_{\ell+1}, y_{\ell+2}, \ldots, y_q)$$

$\text{Distrib. } \mu_{\ell}$

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$$(y_1, \ldots, y_{\ell}, y_{\ell+1}, y_{\ell+2}, \ldots, y_q)$$

$(y_{\ell+2}, \ldots, y_q)$ are distributed identically in both cases → can be dropped
Coupling $\mu_{\ell+1}$ and $\mu_\ell$

$\text{Distrib. } \mu_{\ell+1} \Rightarrow (x_1, \ldots, x_\ell, x_{\ell+1})$

$\text{EM}_{P_1, \ldots, P_r}^{(k_0, \ldots, k_r)} \Rightarrow (y_1, \ldots, y_\ell, y_{\ell+1})$

$\text{Distrib. } \mu_\ell \Rightarrow (x_1, \ldots, x_\ell, u_{\ell+1})$

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\Rightarrow \text{can be dropped}
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Coupling $\mu_{\ell+1}$ and $\mu_{\ell}$

- we will define the second EM cipher (keys and permutations) as a function of the first one in order to have $Y = Z$ with high probability
- first, we choose exactly the same keys
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Coupling $\mu_{\ell+1}$ and $\mu_\ell$

- we will define the permutations $P'_i$ so that $Y = Z$ with high probability.
- first, we define $P'_i(\cdot) = P_i(\cdot)$ on all points encountered during the encryption of $x_1, \ldots, x_\ell$.

This implies $y_1 = z_1, \ldots, y_\ell = z_\ell$. 
Coupling $\mu_{\ell+1}$ and $\mu_{\ell}$

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- first, we define $P'_i(\cdot) = P_i(\cdot)$ on all points encountered during the encryption of $x_1, \ldots, x_\ell$

$Y = (y_1, \ldots, y_\ell, y_{\ell+1})$
$Z = (z_1, \ldots, z_\ell, z_{\ell+1})$
Indistinguishability

The indistinguishability proof

Coupling $\mu_{\ell+1}$ and $\mu_{\ell}$

it remains to equate $y_{\ell+1}$ and $z_{\ell+1}$

let $x_{\ell+1}^i$, resp. $u_{\ell+1}^i$ denote the input to $P_i$, resp $P'_i$, while encrypting $x_{\ell+1}$, resp. $u_{\ell+1}$

recall: the permutations $P_i$ and $P'_i$ must satisfy the equations $P_i(a_{i,j}) = b_{i,j}$

we say $x_{\ell+1}^i$, resp. $u_{\ell+1}^i$ is free if it is different from all $a_{i,j}$'s, $j \in [1, q]$
Coupling $\mu_{\ell+1}$ and $\mu_\ell$

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- let $x_{\ell+1}^i$, resp. $u_{\ell+1}^i$ denote the input to $P_i$, resp $P'_i$, while encrypting $x_{\ell+1}$, resp. $u_{\ell+1}$
- recall: the permutations $P_i$ and $P'_i$ must satisfy the equations $P_i(a_{i,j}) = b_{i,j}$
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- it remains to equate $y_{\ell+1}$ and $z_{\ell+1}$
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- let $x_{\ell+1}^i$, resp. $u_{\ell+1}^i$ denote the input to $P_i$, resp $P'_i$, while encrypting $x_{\ell+1}$, resp. $u_{\ell+1}$
- recall: the permutations $P_i$ and $P'_i$ must satisfy the equations $P_i(a_{i,j}) = b_{i,j}$
- we say $x_{\ell+1}^i$, resp. $u_{\ell+1}^i$ is free if it is different from all $a_{i,j}$'s, $j \in [1, q]$
Indistinguishability

The indistinguishability proof

Coupling $\mu_{\ell+1}$ and $\mu_\ell$

we proceed iteratively for $i = 1..r$ as follows:

- if $u_{\ell+1}^i$ is not free, then $P'_i(u_{\ell+1}^i)$ is imposed by the equations $P'_i(a_{i,j}) = b_{i,j}$
- if $u_{\ell+1}^i$ is free but $x_{\ell+1}^i$ is not, we define $P'_i(u_{\ell+1}^i)$ uniformly at random among possible values
- if $u_{\ell+1}^i$ and $x_{\ell+1}^i$ are both free, we define

$$P'_i(u_{\ell+1}^i) = P_i(x_{\ell+1}^i)$$

$\rightarrow$ successful coupling, the subsequent outputs remain equal
Coupling $\mu_{\ell+1}$ and $\mu_\ell$

We have $Y \neq Z$ only if we fail to couple at all rounds $i = 1, \ldots, r$.

Probability to fail to couple at round $i$
(given that it failed at rounds 1, \ldots, $i-1$):
Since $x_{\ell+1}^i$ and $u_{\ell+1}^i$ are randomized by key $k_{i-1}$, and since $|(a_{i,j})| = q$, the probability that $x_{\ell+1}^i$ or $u_{\ell+1}^i$ is not free is at most $2q/N$.

Hence, the probability to fail to couple at all $r$ rounds and to have $Y \neq Z$ at the output of the two EM ciphers is:

$$\Pr[Y \neq Z] \leq \left(\frac{2q}{N}\right)^r.$$
Coupling $\mu_{\ell+1}$ and $\mu_\ell$

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(given that it failed at rounds $1, \ldots, i - 1$):
Since $x^i_{\ell+1}$ and $u^i_{\ell+1}$ are randomized by key $k_{i-1}$, and since $|(a_{i,j})| = q$, the probability that $x^i_{\ell+1}$ or $u^i_{\ell+1}$ is not free is at most $2q/N$.

Hence, the probability to fail to couple at all $r$ rounds and to have $Y \neq Z$ at the output of the two EM ciphers is:

$$\Pr[Y \neq Z] \leq \left(\frac{2q}{N}\right)^r.$$
Concluding the proof

By the coupling lemma

$$\|\mu_{\ell+1} - \mu_\ell\| \leq \Pr[Y \neq Z] \leq \left(\frac{2q}{N}\right)^r.$$ 

Hence:

$$\|\mu_q - \mu_0\| \leq \sum_{\ell=0}^{q-1} \|\mu_{\ell+1} - \mu_\ell\| \leq 2^r \frac{q^{r+1}}{N^r}.$$ 

which gives the upper bound on advantage on any NCPA distinguisher.
Concluding the proof

By the coupling lemma

\[ \|\mu_{\ell+1} - \mu_{\ell}\| \leq \Pr[Y \neq Z] \leq \left(\frac{2q}{N}\right)^r. \]

Hence:

\[ \|\mu_q - \mu_0\| \leq \sum_{\ell=0}^{q-1} \|\mu_{\ell+1} - \mu_{\ell}\| \leq 2^r \frac{q^{r+1}}{N^r}. \]

which gives the upper bound on advantage on any NCPA distinguisher.
From NCPA to CCA security

We use the following “two weak make one strong” composition theorem:

**Theorem ([MPR07])**

\[\text{Let } E \text{ and } F \text{ be two NCPA-secure block ciphers, with the same domain and resp. key spaces } K_E \text{ and } K_F. \text{ Then } E \circ F^{-1} \text{ is a CCA-secure block cipher with key space } K_E \times K_F.\]

The IEM cipher with 2\(r\) rounds is the composition of 2 IEM ciphers with \(r\) rounds (splitting the key \(k_r = k'_r \oplus k''_r\)):

\[
x \oplus k_0 \rightarrow P_1 \rightarrow P_r \rightarrow P_{r+1} \rightarrow \ldots \rightarrow P_{2r} \oplus k_{2r} \rightarrow y
\]
From NCPA to CCA security

We use the following “two weak make one strong” composition theorem:

**Theorem ([MPR07])**

Let $E$ and $F$ be two NCPA-secure block ciphers, with the same domain and resp. key spaces $\mathcal{K}_E$ and $\mathcal{K}_F$. Then $E \circ F^{-1}$ is a CCA-secure block cipher with key space $\mathcal{K}_E \times \mathcal{K}_F$.

The IEM cipher with $2r$ rounds is the composition of 2 IEM ciphers with $r$ rounds (splitting the key $k_r = k'_r \oplus k''_r$):

![Diagram of IEM cipher with 2r rounds](attachment:image.png)
From NCPA to CCA security

Theorem

For any CCA $D$ making at most $q$ queries to each oracle, the distinguishing advantage against the IEM with $r$ rounds ($r$ even) is at most

$$\mathcal{O}\left(\frac{q^{r/2+1}}{N^{r/2}}\right) = \mathcal{O}\left(\frac{q^{r+2}}{N^{r}}\right).$$

→ security up to $\mathcal{O}(N^{r/(r+2)})$ queries.

New result [CS14]: in fact, security up to $\mathcal{O}(N^{r/(r+1)})$ queries as well.
From NCPA to CCA security

Theorem

For any CCA $D$ making at most $q$ queries to each oracle, the distinguishing advantage against the IEM with $r$ rounds ($r$ even) is at most

$$O\left(\frac{q^r}{N^{r/2}}\right) = O\left(\frac{q^{r+2}}{N^r}\right).$$

→ security up to $O(N^{r/(r+2)})$ queries.

New result [CS14]: in fact, security up to $O(N^{r/(r+1)})$ queries as well.
Extensions and open problems

- Results can be extended to the case where the \((r + 1)\) round keys are \(r\)-wise independent, e.g.:

```
\begin{align*}
  x \rightarrow P_1 & \rightarrow P_2 & \rightarrow \ldots & \rightarrow P_r \\
  k_1 & \rightarrow k_1 & \rightarrow k_2 & \rightarrow k_2 & \rightarrow k_r & \rightarrow k_r \\
\end{align*}
```

- What about the single-key IEM (all round keys equal)?
  - Current conjecture: similar bounds to the “independent round keys” case
Extensions and open problems

- Results can be extended to the case where the \((r + 1)\) round keys are \(r\)-wise independent, e.g.:

  \[
  \begin{array}{cccccccc}
  x & \rightarrow & P_1 & \oplus & k_1 & \rightarrow & P_2 & \oplus & k_1 & \rightarrow & P_{r-1} & \oplus & k_r & \rightarrow & y
  \end{array}
  \]

- What about the single-key IEM (all round keys equal)?
  Current conjecture: similar bounds to the “independent round keys” case
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A tweakable block cipher (TBC) is a family of block ciphers indexed by a tweak $t \in \mathcal{T}$:

$$\tilde{E} : \mathcal{T} \times \mathcal{K} \times \mathcal{M} \rightarrow \mathcal{M}$$

The tweak is a public parameter (under the control of the adversary in the security model).

Introduced by Liskov, Rivest, and Wagner at CRYPTO 2002 [LRW02].
Tweakable block ciphers: definition

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The original [LRW02] construction

Liskov et al. proposed the following construction of a TBC from an existing blockcipher $E$:

$h$ is an $\varepsilon$–AXU$_2$ function: $\Pr_h[h(x) \oplus h(x') = y] \leq \varepsilon$.

[LRW02] proved security (against CCA adversaries) up to $O(2^{n/2})$ queries ($n$ is the block size of $E$)
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[LRW02] proved security (against CCA adversaries) up to $O(2^{n/2})$ queries ($n$ is the block size of $E$)
The [LST12] construction

At CRYPTO 2012, Landecker et al. extended the LRW construction as follows:

\[ E_{k_1} \] 
\[ h_1 \] 
\[ x \] 
\[ t \] 
\[ h_2 \] 
\[ E_{k_2} \] 
\[ y \]

[LST12] proved security (against CCA adversaries) up to \( O(2^{2n/3}) \) queries.
Interlude: tweakable block ciphers

Extension to $r$ rounds

\[ x \xrightarrow{E_{k_1}} h_1 \xrightarrow{E_{k_2}} h_2 \xrightarrow{E_{k_r}} y \]

\[ x \xrightarrow{P_1} k_1 \xrightarrow{P_2} k_2 \xrightarrow{P_r} y \]

\[ \sim \]
Extension to $r$ rounds

$$x \xrightarrow{t} h_1 E_{k_1} h_2 E_{k_2} \cdots \cdots \xrightarrow{t} h_r E_{k_r} y$$

$$x \xrightarrow{k_1} P_1 k_1 \xrightarrow{k_2} P_2 k_2 \cdots \cdots \xrightarrow{k_r} P_r y$$
Extension to $r$ rounds

For this TBC construction, one can prove results similar to the ones for the IEM cipher [LS13]:

- secure against NCPA distinguishers up to $O(2^{rn/(r+1)})$ queries
- secure against CCA distinguishers up to $O(2^{rn/(r+2)})$ queries
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From indistinguishability to indifferentiability

Previous results state that the IEM cipher is a (strong) pseudorandom permutation (in the random permutation model)
≡ usual single, secret key security model

What about related-, known- or chosen-key attacks?
→ prove the IEM is indifferentiable from an ideal cipher

Ideal cipher: draw an independent random permutation for each key
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Ideal cipher: draw an independent random permutation for each key
A word on the ideal cipher model

- the pseudorandomness security notion for a block cipher is sufficient to prove the security of a lot of applications (encryption modes and MACs)
- however, sometimes it is not sufficient (e.g. for block cipher-based hash functions like Davies-Meyer mode)
- ideally, one expects that a good block cipher “behaves” as an independent random permutation for each key
  → ideal cipher model
- similar to the random oracle model for a hash function
- warning: instantiation problems as well (no concrete block cipher can be proved to be an ideal cipher in any reasonable sense)
- though we cannot prove that a block cipher behaves as an ideal cipher in the standard model, we can prove results in idealized models (e.g. the Random Permutation Model that we already used for the IEM cipher)
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Indifferentiability: definition

Definition

A construction $C^F$ (here, the IEM cipher $EM^{P_1,\ldots,P_r}$) using an ideal primitive $F$ (here, random permutations $P_1,\ldots, P_r$) is said indifferentiable from an ideal primitive $G$ (here, an ideal cipher $E$) if there exists a polynomial time simulator $S$ with access to $G$ such that the two systems $(C^F, F)$ and $(G, S^G)$ are indistinguishable.
Indifferentiability: definition

The answers of the simulator $S$ must be:

- **coherent** with answers the distinguisher can obtain directly from $E$
- **close in distribution** to the answers of a random permutation

NB: The distinguisher specifies the key and the plaintext/ciphertext when querying $E^{P_1,\ldots,P_r}$ or $E$.  

---

**Diagram:**

- Boxes labeled $E$, $P_1$, $\ldots$, $P_r$, $D$, and $\sim$ indicating the simulator.
- Edges denote input and output relationships.
- $(K, x/y)$ is input, and $0/1$ is output.
Composition theorem

Usefulness of indifferentiability: composition theorem

**Theorem**

*If a cryptosystem $\Gamma$ is secure when used with an ideal primitive $G$, and if $C^F$ is indifferentiable from $G$, then $\Gamma$ is also secure when used with $C^F$.***

**Sketch of the proof:**

- Assume $C^F$ is indifferentiable from $G$
- Assume there is an attacker $A$ with advantage $\varepsilon$ against some cryptosystem $\Gamma$ using the construction $C^F$
- Then one can consider the simulator $S$ ensured by indifferentiability
- Combining $A$ and $S$, one obtains a new attacker $A'$ against cryptosystem $\Gamma$ used with $G$ with advantage $\approx \varepsilon$, a contradiction
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Indifferentiability of the IEM cipher

Independent round keys fails

This is not indifferentiable from an ideal cipher with key space \( \{0, 1\}^{(r+1)n} \) because of the following distinguisher:

- fix a non-zero constant \( c \in \{0, 1\}^n \)
- choose an arbitrary \( x \in \{0, 1\}^n \) and \( k_0 \in \{0, 1\}^n \)
- define \( x' = x \oplus c \) and \( k'_0 = k_0 \oplus c \)
- let \( K = (k_0, k_1, \ldots, k_r) \) and \( K' = (k'_0, k_1, \ldots, k_r) \)
- then \( EM(K, x) = EM(K', x') \)
- this holds only with negligible probability for an ideal cipher
Proving indifferentiability for key-alternating ciphers

Independent keys leave too much “freedom” to the adversary.

Two ideas to solve the problem:

1. add a key schedule, and put some cryptographic assumption on it
   ⇒ Andreeva et al. CRYPTO 2013 [ABD⁺13]

2. restrain the key space and correlate the round keys, e.g. \((k, k, \ldots, k)\)
   ⇒ Lampe and Seurin 2013 (preprint)
The [ABD$^+$13] result

The key-derivation function is modeled as a random oracle from $\{0, 1\}^\ell$ to $\{0, 1\}^n$ (that the adversary queries in a black-box way)

$\rightarrow$ indifferentiable from an ideal cipher with $\ell$-bit keys for $r = 5$ ([ABD$^+$13] gives attacks up to 3 rounds)

The assumption about the key derivation is very strong and far from concrete designs (the key-schedule is often invertible)
The \([ABD^{+}13]\) result

The key-derivation function is modeled as a random oracle from \(\{0,1\}^\ell\) to \(\{0,1\}^n\) (that the adversary queries in a black-box way)

\[ K \xrightarrow{H} P_1 \xrightarrow{H} P_2 \xrightarrow{H} P_r \rightarrow y \]

\(\rightarrow\) indifferentiable from an ideal cipher with \(\ell\)-bit keys for \(r = 5\) ([ABD^{+}13] gives attacks up to 3 rounds)

The assumption about the key derivation is very strong and far from concrete designs (the key-schedule is often invertible)
Our approach

We consider the IEM with a single key:

\[ x \xrightarrow{k} P_1 \xrightarrow{k} P_2 \xrightarrow{\ldots} P_r \xrightarrow{k} y \]

The trivial attack on independent keys does not apply → is it indiff. from an ideal cipher for sufficiently many rounds?

Main Result

The single-key IEM with \( r = 12 \) rounds is indifferentiable from an ideal cipher with \( n \)-bit blocks and \( n \)-bit keys

Also holds when using invertible permutations \( \gamma_i \) for the key derivation (no cryptographic assumption needed).
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\[
\begin{align*}
x & \xrightarrow{k} P_1 \xrightarrow{k} P_2 \ldots \xrightarrow{k} P_r & \rightarrow y
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A simple attack for 1 round

The distinguisher $D$ proceeds as follows:

- query $P_1(a) = b$ for an arbitrary $a$
- choose a random key $k$ and define $x = a \oplus k$
- query $E(k, x) = y$ and check whether $y = b \oplus k$ (*)

Then:

- when $D$ interacts with a real EM cipher, (*) always holds
- when $D$ interacts with $(E, S^E)$, (*) holds only with negligible probability since $S$ cannot guess $k$ when answering the query $P_1(a)$
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An attack for 3 rounds

One can (easily) find \((x, x', x'', x''')\), \((y, y', y'', y''')\) and \((k, k', k'', k''')\) such that \(y = EM(P_1, P_2, P_3)(k, x)\), etc. and:

\[
\begin{align*}
    k \oplus k' \oplus k'' \oplus k''' &= 0 \\
    x \oplus x' \oplus x'' \oplus x''' &= 0 \\
    y \oplus y' \oplus y'' \oplus y''' &= 0
\end{align*}
\]

This can be showed to be hard for an ideal cipher.
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The simulator must return answers that are coherent with what the distinguisher can obtain from the ideal cipher $E$, i.e.:

$EM^{P_1,\ldots,P_{12}}(k, x) = E(k, x)$

For this, the simulator must adapt at least one permutation to “match” what is given by the ideal cipher
The simulator detects and completes “partial chains” = two adjacent queries $P_i(x_i) = y_i$ and $P_{i+1}(x_{i+1}) = y_{i+1}$

For any partial chain the key is uniquely defined: $k = y_i \oplus x_{i+1}$

When a partial chain is detected, the simulator completes the missing permutation values randomly, except for one particular permutation which is “adapted” to match the ideal cipher.
How the simulator works

- The simulator only detects partial chains at very specific places:
  - External chains \((P_1, P_2, P_{11}, P_{12})\) that matches the ideal cipher \(E\)
  - Central chains \((P_6, P_7)\)

- An external chain can be created only if the distinguisher has made the corresponding query to \(E\) → only \(q\) of them will be completed, which avoids an recursive blow-up of the simulator.
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- the simulator uses specific permutations to adapt chains: $P_4$ and $P_9$
- main difficulty: show that the simulator can always adapt (i.e. the permutation has not already been defined on the point needed for adaptation)
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Open problems

The indifferentiability proof requires 12 rounds, but the best attack is only on 3 rounds.

Conjecture

The single-key IEM with $3 < r < 12$ rounds is indifferentiable from an ideal cipher with $n$-bit keys.

$r = 4$ may well be sufficient.
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Conclusion

Summary of results about the IEM cipher:

- pseudorandomness: the IEM cipher with $r$ rounds is indistinguishable from a random permutation up to $\mathcal{O}(N^r/(r+1))$ queries
- indifferentiability: the single-key IEM cipher with 12 rounds is indifferentiable from an ideal cipher with $n$-bit keys

Interpretation of the results:

- shows that the general strategy of building block ciphers from SPNs is sound and may even yield something close to an ideal cipher
- says little about concrete block ciphers: e.g. the permutations $P_1, \ldots, P_{10}$ of AES-128 are too simple
- heuristic insurance for e.g. an IEM cipher where the $P_i$’s are instantiated with AES used with fixed keys
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Thanks for your attention!
Comments or questions?
Elena Andreeva, Andrey Bogdanov, Yevgeniy Dodis, Bart Mennink, and John P. Steinberger.

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